ON THE THEORY OF STRONG EXPLOSION IN HEAT-CONDUCTING GAS

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In 1946, Sedov gave the exact solution to the problem of strong explosion with plane, cylindrical and spherical shock waves [1]. This solution was obtained for the case of an ideal gas, without friction or heat conduction. As a consequence of these assumptions, there appeared a singularity in the behavior of the solution in the neighborhood of the center of the explosion, where, as is well-known, heat conduction plays the dominating role [2].

The main purpose of the present paper is to use the method of inner and outer asymptotic expansions to construct a solution that is uniformly valid for the entire flow field, including the part near the center. More precisely, the problem consists of finding the leading term of the inner asymptotic expansion by matching it to the solution of Sedov, since the latter is the leading term of the outer asymptotic expansion.

1. The defining parameters for the strong explosion problem (without counterpressure) in a viscous heat-conducting gas are: density of the gas in the undisturbed region ρ_∞, energy E_0 given off by the explosion, and the proportionality constant C in the relation between viscosity coefficient and the specific enthalpy

$$\mu = \mathrm{c}\cosh \tag{1.1}$$

The dimensions of these quantities are as follows:

$$[\rho_\infty] = \frac{M}{L^3}, \qquad [E_0] = \frac{ML^\nu}{T^2}, \qquad [C] = \frac{MT}{L^3} \tag{1.2}$$

Here M, L, T are the corresponding symbols for mass, length, and time, while $\nu = 0, 1, 2$ correspond to plane, cylindrical, and spherical geometries. From the defining parameters, the problem may contain the combinations, with dimensions of time and length, respectively,

$$t^* = \frac{C}{\rho_\infty}, \qquad l^* = C^{\frac{2}{3+\nu}} E^{\frac{1}{3+\nu}} \rho_\infty^{-\frac{3}{3+\nu}} \tag{1.3}$$

Using these quantities as unit quantities, we introduce the dimensionless variables

$$t^\circ = t / t^*, \qquad y^\circ = y / l^* \tag{1.4}$$

The unknown functions of the problem are the velocity v, pressure p, density ρ, and specific enthalpy h. We define their corresponding dimensionless values

$$v^\circ = \frac{v}{l^* / t^*}, \quad p^\circ = \frac{p}{\rho_\infty (l^* / t^*)^2}, \quad \rho = \frac{\rho}{\rho_\infty}, \quad h = \frac{h}{(l^* / t^*)^2} \tag{1.5}$$

The system of Navier-Stokes equations for arbitrary one-dimensional motion of a viscous heat-conducting gas, for the case (1.1) in terms of the dimensionless variables (1.4) and (1.5), is (*)

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y}\right) + \frac{\partial p}{\partial y} = \frac{\partial}{\partial y}\left(\frac{4}{3}\frac{\partial v}{\partial y} - \frac{2}{3}\nu\frac{v}{y}\right) + 2\nu\frac{h}{y}\left(\frac{\partial v}{\partial y} - \frac{v}{y}\right)$$

$$\rho\left(\frac{\partial h}{\partial t} + v\frac{\partial h}{\partial t}\right) = \frac{\partial p}{\partial t} + v\frac{\partial p}{\partial y} + \frac{1}{\sigma}\frac{\partial}{\partial y}\left(hy^\nu\frac{\partial h}{\partial y}\right) + 2h\left[\left(\frac{\partial v}{\partial y}\right)^2 + \nu\left(\frac{v}{y}\right)^2\right] - \frac{2}{3}h\left(\frac{\partial v}{\partial y} + \nu\frac{v}{y}\right)^2$$

$$\frac{\partial}{\partial t}(\rho y^\nu) + \frac{\partial}{\partial y}(\rho v y^\nu) = 0, \qquad p = \frac{\gamma - 1}{\gamma}\rho h \qquad \left(\gamma = \frac{c_p}{c_v}\right) \tag{1.6}$$

Here c_p and c_v are the specific heats of the gas, and σ is the Prandtl number.

For what follows, it would be convenient to change the independent variables y and t of rectangular, cylindrical, or spherical coordinates to the Lagrangian variables ψ and t, which, because of the continuity equation can be given by the relations

1.7
$$\frac{\partial}{\partial y} = \rho y^\nu \frac{\partial}{\partial \psi} \tag{1.7}$$

The Navier-Stokes system in the new independent variables becomes

$$\rho\frac{\partial v}{\partial t} + \rho y^\nu \frac{\partial p}{\partial \psi} = \rho y^\nu \frac{\partial}{\partial \psi}\left[h\left(\frac{4}{3}\rho y^\nu\frac{\partial v}{\partial \psi} - \frac{2}{3}\nu\frac{v}{y}\right)\right] + 2\nu\frac{h}{y}\left(\rho y^\nu\frac{\partial v}{\partial \psi} - \frac{v}{y}\right)$$

$$\rho\frac{\partial h}{\partial t} = \frac{\partial p}{\partial t} + \frac{\rho}{\sigma}\frac{\partial}{\partial \psi}\left(y^{2\nu}\rho h\frac{\partial h}{\partial \psi}\right) + 2h\left[\left(\rho y^\nu\frac{\partial v}{\partial \psi}\right)^2 + \nu\left(\frac{v}{y}\right)^2\right] - \frac{2}{3}h\left(\rho y^\nu\frac{\partial v}{\partial \psi} + \nu\frac{v}{y}\right)^2$$

$$\rho y^\nu\frac{\partial y}{\partial \psi} = 1, \qquad \frac{\partial y}{\partial t} = v, \qquad p = \frac{\gamma - 1}{\gamma}\rho h \tag{1.8}$$

The problem reduces to determining the leading terms of the asymptotic solution of the system (1.8) as $t \to \infty$, which satisfies the initial and boundary conditions.

2. Equations (1.8) are satisfied by the asymptotic solution

*) From here on, the superscript on the dimensionless variables will be dropped for simplicity.

$$y = a_0 t^{\frac{2}{3+\nu}}[Y_0(n) + O(t^{-\varkappa})], \qquad v = \frac{4a_0}{(3+\nu)(\gamma+1)} t^{-\frac{1+\nu}{3+\nu}}[V_0(n) + O(t^{-\varkappa})]$$

$$p = \frac{8a_0^2}{(3+\nu)^2(\gamma+1)} t^{-2\frac{1+\nu}{3+\nu}}[P_0(n) + O(t^{-\varkappa})], \qquad \rho = \frac{\gamma+1}{\gamma-1}[R_0(n) + O(t^{-\varkappa})]$$

$$h = \frac{8a_0^2\gamma}{(3+\nu)^2(\gamma+1)^2} t^{-2\frac{1+\nu}{3+\nu}}[H_0(n) + O(t^{-\varkappa})] \tag{2.1}$$

where only the leading terms are given, (which are represented by the Sedov solution). The terms of the order $t^{-\varkappa}$ are neglected (the value of $\varkappa > 0$ will be determined later). In the above solution a_0 is a constant, and the independent variable

$$n = \frac{1+\nu}{a_0^{1+\nu}} t^{-2\frac{1+\nu}{3+\nu}} \psi \tag{2.2}$$

The system of ordinary differential equations for the functions $F_0(n)$, which is obtained by substituting (2.1) into (1.8), is the well-known system of equations of one-dimensional iso-energetic motion of an ideal gas [2].

After a simple transformation, this system may be written as (*)

$$\begin{gathered} 2P_0' = 2nV_0' + V_0, \qquad (nP_0 / R_0^{\gamma})' = 0, \quad P_0 = R_0 H_0 \\ (1+\nu)\frac{\gamma+1}{\gamma-1} R_0 Y_0^{\nu} Y_0' = 1, \quad (1+\nu) n Y_0' - Y_0 + \frac{2}{\gamma+1} V_0 = 0 \end{gathered} \tag{2.3}$$

The solution of these equations must satisfy the boundary conditions on the surface of the shock wave, which for the functions $F_0(n)$ given by (2.1), will be

$$F_0(1) = 1 \tag{2.4}$$

Also, it must satisfy the condition of constant total energy in the disturbed region (**)

$$\int_0^1 \left(\frac{P_0}{R_0} + V_0^2\right) dn = \frac{(3+\nu)^2(\gamma+1)^2(1+\nu)}{2^{3+\nu}\pi^k a_0^{3+\gamma}} E \qquad \begin{pmatrix} k=0 & \text{for } \nu = 0 \\ k=1 & \text{for } \nu = 1,2 \end{pmatrix} \tag{2.5}$$

For subsequent use, it is sufficient to have only the approximate representation of the exact solution of (2.3) to (2.5) in a small neighborhood of the center of the explosion. In this region (as $n \to 0$), the solution has the form (2.6)

$$Y_0 = Y_{00} n^{\frac{\gamma-1}{\gamma(1+\nu)}}[1 + O(n^{\alpha})], \quad V_0 = V_{00} n^{\frac{\gamma-1}{\gamma(1+\nu)}}[1 + O(n^{\alpha})] \qquad \left(\alpha = \frac{\gamma-1}{\gamma(1+\nu)}\right)$$

$$P_0 = P_{00}[1 + O(n^{\alpha})], \quad R_0 = R_{00} n^{\frac{1}{\gamma}}[1 + O(n^{\alpha})], \quad H_0 = H_{00} n^{-\frac{1}{\gamma}}[1 + O(n^{\alpha})]$$

Here the constants, by (2.3), are connected by the relations

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*) Here and below, the prime indicates the derivative.

**) Here E is the dimensionless energy of the explosion

$$E = E_0 t^{*2} / \rho_{\infty} l^{*3+\nu}$$

$$Y_{00}=\left(\frac{\gamma}{\gamma+1}P_{00}^{-\frac{1}{\gamma}}\right)^{\frac{1}{1+\nu}},\qquad R_{00}=P_{00}^{\frac{1}{\gamma}}$$

$$V_{00}=\frac{\gamma+1}{2\gamma}\left(\frac{\gamma}{\gamma+1}P_{00}^{-\frac{1}{\gamma}}\right)^{\frac{1}{1+\nu}},\qquad H_{00}=P_{00}^{\frac{\gamma-1}{\gamma}} \tag{2.7}$$

i.e. they are expressed in terms of the constant P_{00}, which, just as a_0 in (2.1), is known from the exact solution [2].

3. To determine the asymptotic expansions valid in the inner region of the flow, it is necessary to use the matching conditions between these expansions and the outer expansions (2.1). Suppose that the independent variable of order of unity for the inner region is

$$N=\psi t^{-2\frac{1+\nu}{3+\nu}+\delta} \tag{3.1}$$

where $\delta > 0$, so that by (2.2), we have

$$n=\frac{1+\nu}{a_0^{1+\nu}}Nt^{-\delta} \tag{3.2}$$

According to the well-known principle of matching of the inner and outer expansions [3], it is necessary to require that the inner limit of the outer expansions (obtained by substituting (3.2) into (2.1) and then passing to the limit $t\to\infty$ for fixed N) to agree with the outer limit of the inner expansions (obtained when $t\to\infty$ with fixed n, i.e. by (3.2), as $N\to\infty$).

As a result of using (2.6), we find the following expressions for these limits, written in terms of the variables of the inner expansions

$$y=a_0Y_{00}\left(\frac{1+\nu}{a_0^{1+\nu}}N\right)^{\frac{\gamma-1}{\gamma(1+\nu)}}t^{\frac{2}{3+\nu}-\delta\frac{\gamma-1}{\gamma(1+\nu)}}[1+O(t^{-\beta})]\qquad\left(\beta=+\delta\frac{\gamma-1}{\gamma(1+\nu)}\right)$$

$$v=\frac{4a_0}{(3+\nu)(\gamma+1)}V_{00}\left(\frac{1+\nu}{a_0^{1+\nu}}N\right)^{\frac{\gamma-1}{\gamma(1+\nu)}}t^{-\frac{1+\nu}{3+\nu}-\delta\frac{\gamma-1}{\gamma(1+\nu)}}[1+O(t^{-\beta})]$$

$$p=\frac{8a_0^2}{(3+\nu)^2(\gamma+1)}P_{00}t^{-2\frac{1+\nu}{3+\nu}}[1+O(t^{-\beta})] \tag{3.3}$$

$$\rho=\frac{\gamma+1}{\gamma-1}R_{00}\left(\frac{1+\nu}{a_0^{1+\nu}}N\right)^{\frac{1}{\gamma}}t^{-\frac{\delta}{\gamma}}[1+O(t^{-\beta})]$$

$$h=\frac{8a_0^2\gamma}{(3+\nu)^2(\gamma+1)^2}H_{00}\left(\frac{1+\nu}{a_0^{1+\nu}}N\right)^{\frac{1}{\gamma}}t^{-2\frac{1+\nu}{3+\nu}+\frac{\delta}{\gamma}}[1+O(t^{-\beta})]$$

To find the value of δ, appearing in the exponents of the expressions (3.1) to (3.3), we shall define the inner region of the flow as the neighborhood of the center of the explosion in which heat-conduction plays the dominant role. Turning to the energy equation (1.8), using the transformation formulas for the independent variables t and N

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t}-\left(2\frac{1+\nu}{3+\nu}-\delta\right)\frac{N}{t}\cdot\frac{\partial}{\partial N},\qquad \frac{\partial}{\partial\psi}=t^{-2\frac{1+\nu}{3+\nu}+\delta}\frac{\partial}{\partial N} \tag{3.4}$$

and substituting into (1.8) Expressions (3.3), we find

$$\delta = \frac{\gamma(1+\nu)}{2(\gamma+\nu)} \tag{3.5}$$

4. In this manner, the form of the inner asymptotic solution is now known, and it can be written as

$$y = t^{\frac{2}{3+\nu} - \frac{\gamma-1}{2(\gamma+\nu)}}[y_0(N) + O(t^{-\lambda})], \qquad v = t^{-\frac{1+\nu}{3+\nu} - \frac{\gamma-1}{2(\gamma+\nu)}}[v_0(N) + O(t^{-\lambda})]$$

$$p = t^{-2\frac{1+\nu}{3+\nu}}[p_0(N) + O(t^{-\lambda})], \qquad \rho = t^{-\frac{1+\nu}{2(\gamma+\nu)}}[\rho_0(N) + O(t^{-\lambda})]$$

$$h = t^{-2\frac{1+\nu}{3+\nu} + \frac{1+\nu}{2(\gamma+\nu)}}[h_0(N) + O(t^{-\lambda})] \tag{4.1}$$

(the quantity $\lambda > 0$ will be determined later).

The conditions for matching these expansions to the outer asymptotic expansions, because of (3.3) and (3.5), are

$$y_0(N) \to a_0 Y_{00}\left(\frac{1+\nu}{a_0^{1+\nu}} N\right)^{\frac{\gamma-1}{\gamma(1+\nu)}}, \quad v_0(N) \to \frac{4a_0}{(3+\nu)(\gamma+1)V_{00}}\left(\frac{1+\nu}{a_0^{1+\nu}} N\right)^{\frac{\gamma-1}{\gamma(1+\nu)}}$$

$$p_0(N) \to \frac{8a_0^2}{(3+\nu)^2(\gamma+1)} P_{00}, \quad \rho_0(N) \to \frac{\gamma+1}{\gamma-1} R_{00}\left(\frac{1+\nu}{a_0^{1+\nu}} N\right)^{1/\gamma} \qquad \text{for } N \to \infty$$

$$h_0(N) \to \frac{8a_0^2\gamma}{(3+\nu)^2(\gamma+1)^2} H_{00}\left(\frac{1+\nu}{a_0^{1+\nu}} N\right)^{-1/\gamma} \tag{4.2}$$

The system of ordinary differential equations, which must be satisfied by the functions $f_0(N)$ in the expansions (4.1), is obtained by substituting these expansions in the original system of equations (1.8).

As a result, we find

$$p_0' = 0, \quad \frac{1}{\sigma}\frac{\gamma}{\gamma-1} p_0 (y_0^{2\nu} h_0')' + \left[2\,\frac{1+\nu}{3+\nu} - \frac{(1+\nu)\gamma}{2(\gamma+\nu)}\right]\left(N h_0' + \frac{h_0}{\gamma}\right) = 0$$

$$\frac{\gamma}{\gamma-1} p_0 y_0^{\nu} y_0' = h_0, \qquad \rho_0 = \frac{\gamma}{\gamma-1}\frac{p_0}{h_0}$$

$$v_0 = \left[\frac{2}{3+\nu} - \frac{\gamma-1}{2(\gamma+\nu)}\right] y_0 - \left[2\,\frac{1+\nu}{3+\nu} - \frac{(1+\nu)\gamma}{2(\gamma+\nu)}\right] N y_0' \tag{4.3}$$

Boundary conditions for these equations, in addition to the asymptotic conditions (4.2), are the obvious symmetry conditions at the center of the explosion

$$y_0(0) = v_0(0) = h_0'(0) = 0 \tag{4.4}$$

The remaining problem is to integrate system (4.3) under these boundary conditions.

5. First of all, we note that according to the first equation of (4.3), the pressure, as is to be expected, is constant across the inner region; and by (4.2)

$$p_0 = \frac{8a_0^2}{(3+\nu)^2(\gamma+1)} P_{00} \tag{5.1}$$

From now on, we let

$$y_0 = g_0^{\frac{1}{1+\nu}} \tag{5.2}$$

From the third equation of (4.3), we have

$$h_0 = \frac{\gamma}{(1+\nu)(\gamma-1)} p_0 g_0' \tag{5.3}$$

Then, integrating the second equation (4.3) with boundary conditions (4.4), we get

$$g_0^m g_0'' + A\left(N g_0' - \frac{\gamma-1}{\gamma} g_0\right) = 0, \qquad m = \frac{2\nu}{1+\nu}$$
$$A = \frac{\gamma-1}{\gamma p_0} \sigma \left[2\,\frac{1+\nu}{3+\nu} - \frac{(1+\nu)\gamma}{2(\gamma+\nu)}\right] \tag{5.4}$$

Thus, the problem is reduced to integrating a single ordinary differential equation of the second order with boundary conditions

$$g_0(0) = 0, \qquad g_0(N) \to a_0^{1+\nu} Y_{00}^{1+\nu} \left(\frac{1+\nu}{a_0^{1+\nu}} N\right)^{\frac{\gamma-1}{\gamma}} \quad \text{for } N \to \infty \tag{5.5}$$

It is not difficult to establish the behavior of the solution to (5.4) and (5.5). The first condition in (5.5), together with the obvious requirement that the first derivative $g_0'(0)$ be finite and non-zero, leads to a unique representation of the desired function in the neighborhood of the singular point $N = 0$

$$g_0(N) = cN\,(1 + a_1 N^\alpha + a_2 N^{2\alpha} + \ldots) \qquad (\alpha = 2 - m) \tag{5.6}$$

Here c is an arbitrary constant, which is to be chosen to satisfy the second condition in (5.5). The remaining coefficients of (5.6) are determined by recurrence relations, which we shall not write down here. In the neighborhood of the point at infinity, Equation (5.4) is satisfied by an asymptotic representation of the form

$$g_0(N) = kN^{\frac{\gamma-1}{\gamma}}\,[1 + b_1 N^{-\beta} + b_2 N^{-2\beta} + \ldots] \qquad \left(\beta = 2 - m\,\frac{\gamma-1}{\gamma}\right) \tag{5.7}$$

The main term of this expansion, as is obvious from the above, corresonds to condition (5.5), consequently, all its coefficients are known.

The integration of Equation (5.5) was carried out numerically. The method used is described in Section 7.

6. The degree of approximation ensured by the outer and inner asymptotic solutions, will be determined by the order of the terms neglected in expansions (2.1) and (4.1), i.e. by the value of the exponents $\varkappa$ and λ in these expansions.

First of all, we note that in the outer flow region, the ratio of the viscous and heat-conduction terms (neglected in (2.3)) to the inertial terms is of the order t^{-1}, as readily shown. Furthermore, the integral of the total flow energy (2.5) was written without considering the contribution of the inner region, and therefore it contains an error, whose order is to be estimated. To obtain the estimate, we consider the integral of the total energy of the gas in the disturbed region $(0, n)$

$$E(n) = \mathrm{const} \int_0^n \left(\frac{P_0}{R_0} + V_0^2 \right) dn \tag{6.1}$$

In this expression, we pass to the inner limit (i.e. we express $E(n)$ in terms of the inner expansion variables using (3.2), and pass to the limit of $t \to \infty$ for fixed N), and using (2.6), we get

$$E(n) \sim t^{\frac{(\gamma-1)(1+\nu)}{2(\gamma+\nu)}} \qquad \text{for } n \to 0 \tag{6.2}$$

Since in the outer flow-region, where $n \sim 1$, the quantity $E(n) \sim 1$ and $\frac{1}{2}(\gamma - 1)(1 + \nu)/(\gamma + \nu) < 1$, then the estimate obtained just determines the degree of approximation ensured by the solution (2.1).

Thus, we have

$$\varkappa = \tfrac{1}{2}(\gamma - 1)(1 + \nu)/(\gamma + \nu) \tag{6.3}$$

In the inner region, the ratio of the viscous and inertial terms (neglected in (4.3)) to the main terms is of the order $t^{-1+\beta'}$, where $\beta' = \frac{1}{2}(1+\nu)/(\gamma+\nu)$, and for the exponent in Expressions (3.3) we have $\beta = \frac{1}{2}(\gamma - 1)(\gamma + \nu)$. Since $\beta < 1 - \beta'$, so in the expansions (4.1) we should set

$$\lambda = \tfrac{1}{2}(\gamma - 1)/(\gamma + \nu) \tag{6.4}$$

7. As an example, the flow parameters in the center region of a point explosion with spherical shock waves ($\nu = 2$), were calculated. The adiabatic exponent and the Prandtl number were assumed to be: $\gamma = 1.4$ and $\sigma = 0.7$ The constants P_{00} and a_0 were taken from the solution in [2]: $a_0 = 1.033$, $P_{00} = 0.3655$.

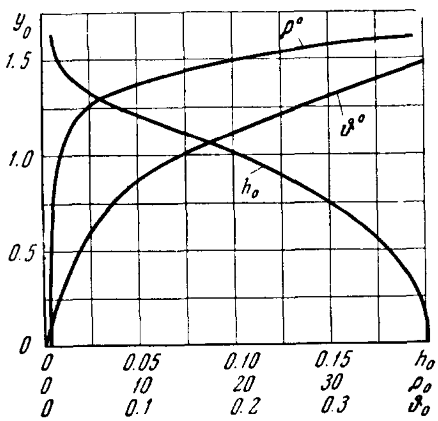
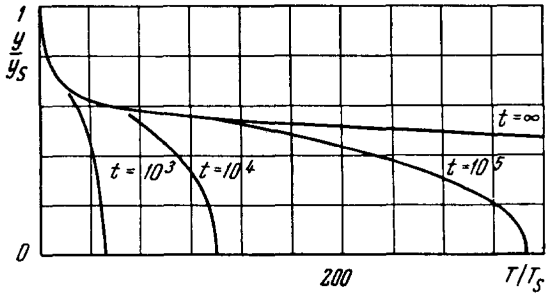


Fig. 1

The numerical integration of Equation (5.4) was carried out with the aid of an electronic computer by the method of successive approximations. (Calculations were made by engineer N.S. Matveeva, to whom the author expresses his thanks.) Without going into details, we present the basic features of this method. The equation for the kth approximation was expressed in the form of a linear equation

$$g'''_{0(k-1)} g''_{0(k)} + A \left(N g'_{0(k)} - \frac{\gamma - 1}{\gamma} g_{0(k)} \right) = 0 \tag{7.1}$$

Fig. 2



For the initial (zeroth) approximation the following linear function was taken:

$$g_{0(0)} = \frac{kN^{*\frac{\gamma-1}{\gamma}}}{N^*} N \tag{7.2}$$

which satisfies the boundary conditions (5.5). The quantity N^* has been chosen sufficiently large. Integration of (7.1) in each of the subsequent approximations (for chosen value of N^*) was carried out by the iteration method. After the necessary number of approximations were completed, the calculations were repeated for a larger value of N^*, and so forth. The integration step (variable in the interval $0, N^*$), the number of iterations, and the value of N^*, were all determined by the requirement that the accuracy be of the order of 0.01%

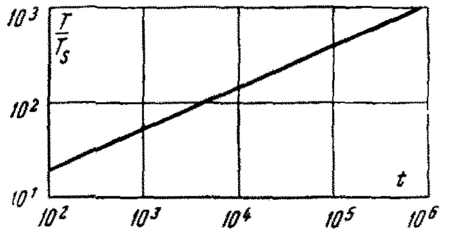


Fig. 3

The computed results for the central region are presented in Fig.1, showing the functions $h_0(y_0)$, $\rho_0(y_0)$ and $v_0(y_0)$ obtained by numerical integration of Equations (5.4). Fig.2 shows the temperature variation in the entire flow field at different instants of time. The temperature is taken relative to its value at the same instant on the shock wave surface, thus

$$\frac{T}{T_s} = \frac{(3+\nu)^2(\gamma+1)^2}{8a_0{}^2\gamma} h_0 t^{\frac{1+\nu}{2(\gamma+\nu)}} \tag{7.3}$$

while the distance from the center of the explosion is taken relative to the instantaneous value of the radius of the wave surface, thus

$$\frac{y}{y_s} = \frac{y_0}{a_0} t^{-\frac{\gamma-1}{2(\gamma+\nu)}} \tag{7.4}$$

As is clear, the curves representing the temperature profiles at different instants of time ($t = 10^3$, 10^4, 10^5) smoothly pass over to an envelope, corresponding to the Sedov solution for the outer region ($t = \infty$).

The variation of the quantity T/T_s at the center of the explosion is shown in Fig.3. We recall that everywhere t denotes the dimensionless time defined by (1.3) and (1.4).

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